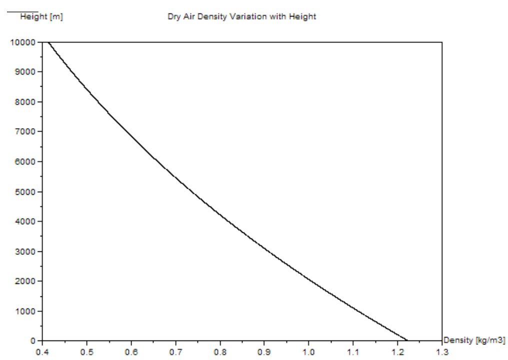
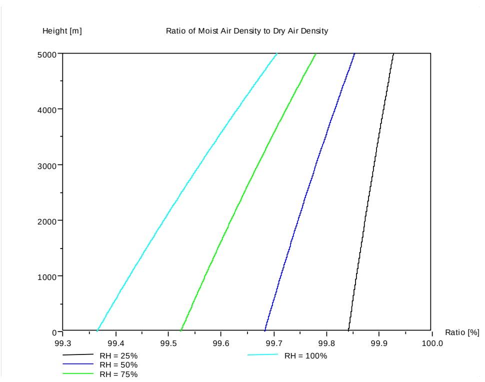
windPRO / ENERGY

Modelling of the Variation of Air Density with Altitude through Pressure, Humidity and Temperature





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Front cover

The front cover shows the air density variation with height – calculated using the U.S. Standard Atmosphere parameters. Also shown is the relative influence of moist air.

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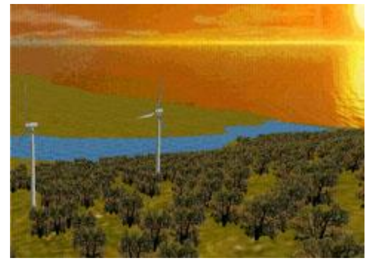
1. Introduction



The air density calculations have been subjected to international standardization, presumably because the air density influences the lifting capacity of aircrafts. The air density varies with altitude and temperature. For wind turbines, the air density is a key parameter when estimating wind energy, as the energy output from the WTGs proportionally depends on this parameter. The estimated power output from the WTG, *P*, is given in the equation below.

$$P = 0.5 \cdot \rho \cdot w^3 \cdot A \cdot C_e \qquad (1)$$

where P is the produced power output [W]
ρ is the air density [kg/m^3]
w is the wind speed [m/s]
A is the area swept by the rotor [m^2]
C_e is the total efficiency of the WTG at the given wind speed

While the energy calculations typically are calculations over a large time scale, the air density must be given as the expected mean density over the period considered. As an alternative to using a constant air density – which is reasonable assumption in most terrains - WindPRO offers a model to take the variations in air density with air temperature and pressure into account. This model could be used in mountainous terrains.

The model for the varying dry air density is mainly based on equations in the US Standard Atmosphere Model from 1976 [1]. Since moist air is less dense than dry air, also a model for including the variation in humidity is also implemented. This is based on equations from M. Salby [2].

The WindPRO model offers the option for the user to set site-specific temperatures, humidity and pressure parameters – thus enabling a detailed air density description for turbines situated with very varying hub-heights.

References

[1] *U.S. Standard Atmosphere, 1976,* U.S. Government Printing Office, Washington, D.C.
[2] Murry L. Salby: *Fundamentals of Atmospheric Physics*, Academic Press, 1996, Elsevier Science

2. Dry Air Density Variation with Altitude

This chapter gives an introduction to the physical and theoretical considerations needed when developing the set of equations describing the variation of dry air with altitude.

Cronalaght Wind Farm, Ireland.

Geopotential Altitude

Modelling of the standard atmosphere is typically done in terms of the geopotential altitude. This is due to a simplification of the equations describing the atmosphere. The idea behind the geopotential altitude is that a small change in geopotential altitude will cause the same change in gravitational potential energy as the same change in geometric altitude at sea level. This is mathematically expressed as:

$$g(Z)dZ = GdH \qquad (1)$$

where $g(Z)$ is the acceleration of gravity (decreases at increasing altitudes)
Z is the geopotential altitude
G is the acceleration of gravity at sea level
dH is the geometric altitude

The value of the gravity varies with height, and is found to follow the equation below

$$\frac{g(Z)}{G} = \left(\frac{E}{Z+E}\right)^2 \qquad (2)$$

where E is the radius of the earth (~6535 km)

Combining the two equations above and integrating yields give an expression for the geopotential altitude.

$$\begin{aligned} dH &= \frac{g}{G}dZ = \left(\frac{E}{Z+E}\right)^2 dZ \\ H &= \int_0^H dH = \int_0^Z \left(\frac{E}{Z+E}\right)^2 dZ = \frac{EZ}{E+Z} \\ Z &= \frac{EH}{E+H} \end{aligned} \qquad (3), (4), (5)$$

The relation between geopotential height and geometric height is used to calculate the table below. It is seen, that the difference between the two altitude measures is marginal for the lower levels of the atmosphere where wind turbines are situated. Thus, in WindPRO we approximate the geopotential height with the geometric height.

Geometric Height, Z *[m]*	*Geopotential Height,* H *[m]*	*Differnce* *%*
100	99.99	0.002
500	499.96	0.008
1000	999.84	0.016
5000	4996.1	0.078
10000	9984.3	0.157
50000	49610	0.796
100000	98451	1.573

Table 1: Relation between geometric and geopotential height

The Perfect Gas Law

The air density may be estimated from the perfect gas law.

$$P \cdot V = n \cdot R_{air} \cdot T \tag{6}$$

where *P* is the pressure [Pa]
V is the volume considered [m^3]
n is the number of moles
R_{air} is the specific gas constant for dry air (287.05 J/(kg·K))
T is the temperature [K]

If the density, ρ_{air}, is defined from the number of molecules in a certain volume, $\rho = n / V$, then the density may now be expressed from the perfect gas law as:

$$\rho_{air} = \frac{P}{R_{air} \cdot T} \tag{7}$$

The specific gas constant for dry air relates to the universal gas constant, R, and the mean molecular weight of air:

$$R_{air} = R / M_{air} \tag{8}$$

where *R* is the universal gas constant = 8.31432 J/(K·mol)
M_{air} is $28.9644 \cdot 10^{-3}$ (kg/mol) (assumed constant up to approximately 86 km altitude)

Temperature Variation in the Atmosphere

Within an atmospheric layer, the temperature variation is approximated as a linear function of the geopotential altitude.

$$T = T_b + L(H - H_b) \tag{9}$$

where *L* is the temperature lapse rate (temperature gradient) – which typically is negative for increasing heights
T_b is the temperature at the base of the layer
H_b is the geopotential altitude at the base level
H is the geopotential height

The Hydrostatic Equation

Hydrostatic modelling of the atmosphere is a reasonable approximation – even if the atmosphere is in motion. This is because the vertical displacements of air and their time derivatives are small compared to the forces in the hydrostatic equation – see M. Salby [1]. The basic hydrostatic equation is

$$\begin{aligned} dP &= -\rho g(Z)dZ \\ &= -\rho G dH \end{aligned} \tag{10}$$

Air Density Calculations with Pressure Changes

Using the hydrostatic equation with the perfect gas law and the stepwise linear temperature variation assumption, the hydrostatic equation yield:

$$dP = -\frac{MG}{R_{air}} \frac{P}{(T_b + L(H - H_b))} dH \tag{11}$$

Integrating the equation gives the relation between base variables and the ones in the specified height [2]

$$\int_{P_b}^{P} \frac{1}{P} dP = -\int_{H_b}^{G} \frac{MG}{R_{air}} \frac{1}{(T_b + L(H - H_b))} dH$$

$$\frac{P}{P_b} = \exp\left[\frac{-GM(H - H_b)}{RT_b}\right] \qquad \text{for } L = 0 \quad \text{and} \qquad (12), (13), (14)$$

$$\frac{P}{P_b} = \left[\frac{T_b + L(H - H_b)}{T_b}\right]^{\frac{\mathrm{GM}}{\mathrm{RL}}} \qquad \text{for } L \neq 0$$

US Standard Atmosphere

The US Standard Atmosphere, published in 1986, holds a model for the development of pressure and density with altitude over the sea level. The model used in WindPRO is based on the specification for the lower 11 km of the atmosphere.

In the US Standard Atmosphere, seven fundamental layers are defined in the lower 86 kilometres of the atmosphere:

h_1 [km]	0	11	20	32	47	51	71
h_2 [km]	11	20	32	47	51	71	84.852
L *(dT/dh)* [K/km]	-6.5	0.0	1.0	2.8	0.0	-2.8	-2.0

The heights are given in geopotential heights. 84.852 km corresponds to a geometric height of 86 km. Standard values of other important parameters are:

Sea level pressure, p_0	101325 N/m2
Sea level temperature, T_0	288.15 K
Hydrostatic constant	34.1631947 K/km

The standard sea level density that is calculated from the settings above is 1.225 kg/m^3. The variation of dry air density and pressure - using the US standard atmosphere parameters - are shown in the Figure 1 and Figure 2 below.

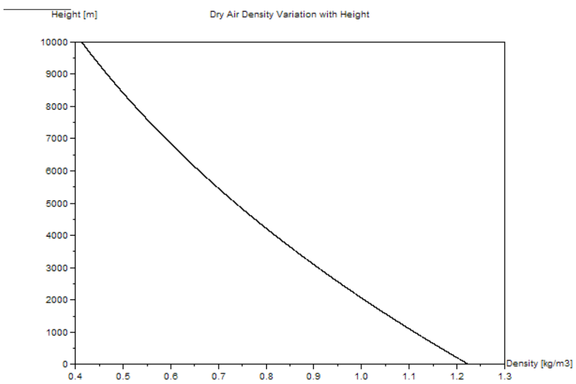


Figure 1: Dry Air Density Variation.

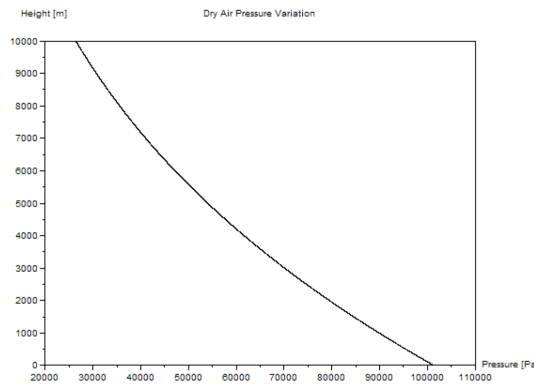


Figure 2: Dry Air Pressure Variation-

References

[1] Murry L. Salby: *Fundamentals of Atmospheric Physics*, Academic Press, 1996, Elsevier Science
[2] Ralph L. Carmichael, *The Hydrostatic Equations*, 2003 (internet note, www.pdas.com/coesa.htm)

3. Density of Moist Air

Even if moist air is less dense than dry air, the water vapor seldom represents more than a few percent of the air mass.

The model for moist air is based on the Dalton law for partial pressures, and is included in WindPRO in order to complete the description of the air density variation. The description is based on M. Salby [1], and is valid for air not condensed.

In WindPRO the user may input three different measures of the vapor:

Brokilde Wind Farm, Denmark.

- Relative humidity
- Dew point temperature
- Specific humidity

The user inputs are – however – always converted into a relative humidity – which is saved with the WindPRO project.

Density of Moist Air – Governing Equation

The density is determined as a mixture of dry air molecules and water vapour molecules

$$\rho_{air} = \frac{P_d(T)}{R_d T} + \frac{P_v(T)}{R_v T} \qquad (1)$$

where ρ_{air} is the density [kg/m^3]
P_d is the partial pressure of dry air [Pa]
P_v is the water vapour partial pressure [Pa]
R_d is the specific gas constant for dry air [J/(kgK)]) = 287.05
R_v is the specific gas constant for water vapour [J/(kgK)] = 461.495
T is the temperature [K]

Saturation Vapor Pressure

In order to calculate the density of moist air, we need to know the saturated vapour pressure. This concept is related to the process of evaporation. Considering a closed container with water and air, the evaporation process will proceed until there are as many molecules returning to the liquid as there are escaping. When this balance is achieved then the vapour is said to be saturated (and the corresponding pressure denoted saturated vapour pressure). When the saturated vapour

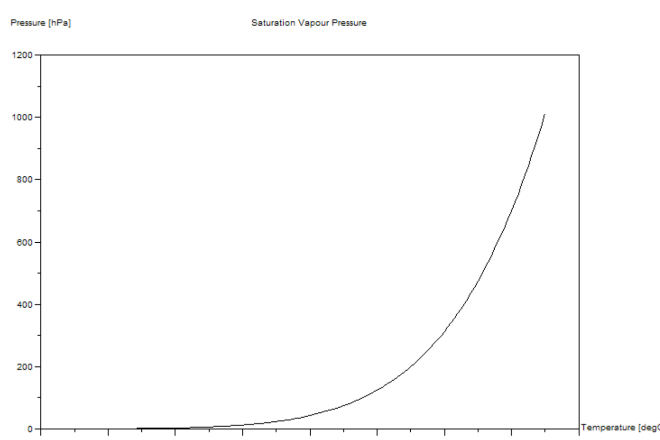


Figure 1: Variation of the saturation vapor pressure.

pressure is equal to the atmospheric pressure, then the liquid is boiling.

In WindPRO, the saturated vapour pressure is calculated using an approximating polynomial, suggested by Herman Wobus. The polynomial was fitted from data from the Smithsonian Meteorological Tables by Roland List (6th edition), and is valid for temperature ranges from -50 °C to 100 °C. A graph of the saturated vapour pressure variation with temperature is found in Figure 1.

$$
\begin{aligned}
&E_s(T) = e_{s0} / p(T)^8 \\
&p(T) = c_0 + T(c_1 + T(c_2 + T(c_3 + T(c_4 + T(c_5 + T(c_6 + T(c_7 + T(c_8 + T \cdot c_9)))))))) \\
&e_{s0} = 6.1078 \\
&c_0 = 0.99999683 \cdot 10^{0} \quad c_1 = -0.90826951 \cdot 10^{-2} \quad c_2 = 0.78736169 \cdot 10^{-4} \\
&c_3 = -0.61117958 \cdot 10^{-6} \quad c_4 = 0.43884187 \cdot 10^{-8} \quad c_5 = -0.29883885 \cdot 10^{-10} \\
&c_6 = 0.21874425 \cdot 10^{-12} \quad c_7 = -0.17892321 \cdot 10^{-14} \quad c_8 = 0.11112018 \cdot 10^{-16} \\
&c_9 = -0.30994571 \cdot 10^{-19}
\end{aligned}
\tag{2}
$$

where T is the temperature in [°C]
Es is the saturated vapour pressure in [mb]

Different Measures of Humidity

The actual vapour pressure is now determined from either the dew point or the relative humidity. If the dew point, T_{dew}, is known, then the actual vapor pressure is simply

$$P_v(T) = E_s(T_{dew}) \tag{3}$$

If the relative humidity (*RH*) is known then the actual vapour pressure simply determined from the definition of the relative humidity. I.e. the relative humidity is simply the ratio of actual vapor pressure to the saturation vapor pressure at a given temperature.

$$P_v(T) = RH \cdot E_s(T) \tag{4}$$

The specific humidity is the relative concentration of vapor. It is defined from the following equation

$$SH = \rho_v / \rho = m_v / m \tag{5}$$

where ρ_v is the absolute concentration of vapor ($\rho_v = 1/v_v$)
v_v is the specific volume of vapor ($v_v = V/m_v$, V is the volume and m_v is the mass of the vapor)
ρ is the density of the air
m_v and m is the masses of vapor and the mixture

In case that the specific humidity is known (*SH*) then the vapor pressure is calculated from (see M. Salby [1]):

$$
\begin{aligned}
&r = SH/(1 - SH) \quad \text{and} \\
&P_v = (r/\varepsilon) \cdot P
\end{aligned}
\tag{6}
$$

where r is the mixing ratio
ε is the ratio of molar weights ≅ 0.622

Calculating the Air Density of Moist Air

The specific gas constant of the mixture of dry and moist air is expressed as:

$$
\begin{aligned}
R_{mix} &= (1-SH)\cdot R_d + SH \cdot R_v \\
&= (1-SH)\cdot R_d + \frac{SH}{\varepsilon}\cdot R_d \\
&= (1+\left(\frac{1}{\varepsilon}-1\right)SH)\cdot R_d
\end{aligned}
\tag{7}
$$

Using this new gas constant with the equations (13) and (14) established in the previous Chapter 2, enables us to calculate the density of moist air. Note, it is an implicit assumption that the gas constant of the mixed gas remains constant through varying heights. This is presumably a reasonable assumption – because the correction due to moisture is very small.

Sample Calculation with US Standard Atmosphere Parameters

In the Figure 2 and Figure 3 the air density variation with height is shown for the U.S. Standard Atmosphere, 1976. Included are also different measures of relative humidity – which are shown to have only a quite small influence.

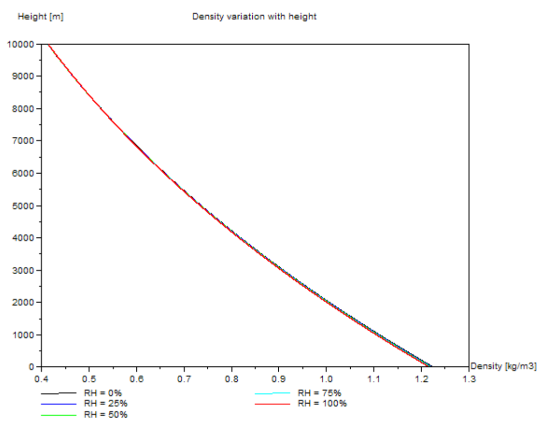


Figure 2: Moist Air Density Variation.

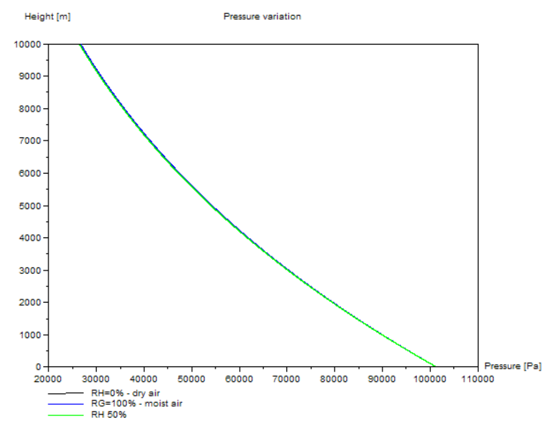


Figure 3: Moist Air Pressure Variation.

In Figure 4 the relative difference between dry air and moist air is shown for the first 5000 meters above the sea level. It is observed that the difference is below 0.7%. Again, the figure is calculated using the US Standard Atmosphere parameters.

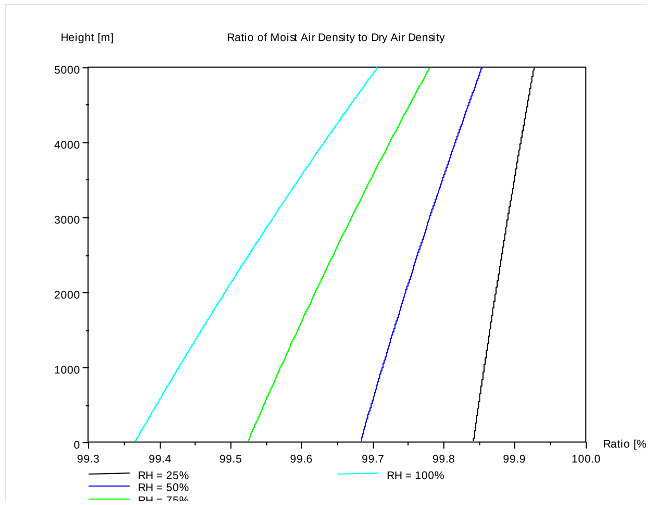


Figure 4: Ratio of Moist Air to Dry Air.

Air Density Variation with Humidity and Pressure (alternative equations)

The *CRC Handbook of Chemistry and Physics* [2] holds an alternative description of the of the moist air density variation with temperature, pressure and vapor pressure. The relation is:

$$D = 1.2929 \cdot (273.13 / T)[(B - 0.7383e)/760] \tag{8}$$

where T is the absolute temperature (degK)
B is the barometric pressure (mmHG)
e is the vapor pressure (mmHG)

This equation (8) is now used to make a sensitivity study of the influence of moist air. This is done by calculating the ratio of moist air density to dry air density. This calculation is shown – for temperatures 5°C - 20 °C in the figures below. From the Figure 5 - Figure 8 it is observed, that the density influence of moist air is less than 1.7 percent for the considered temperature range. Higher temperatures yield a higher influence. For wind turbine applications the air moisture density influence could – in most cases – be ignored (i.e. it is a reasonable assumption to use the dry air properties).

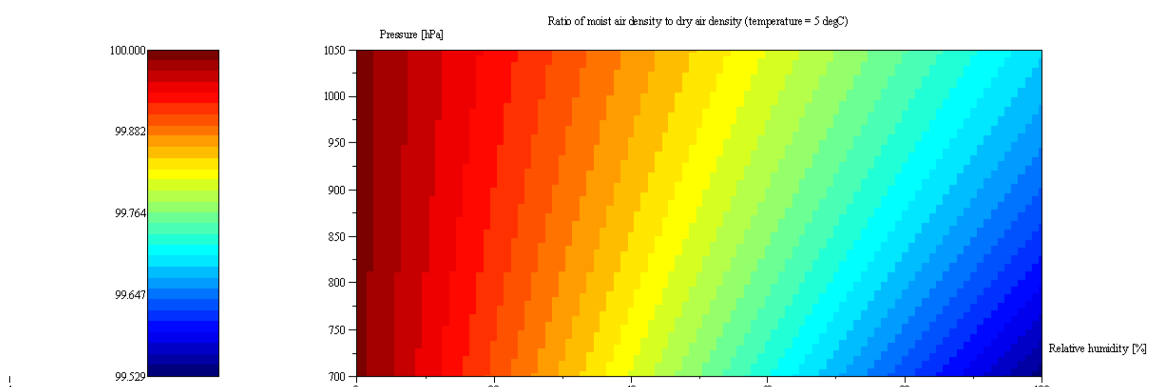


Figure 5: Density Ratio for Temperature = 5 deg C in [%].

Figure 6: Density Ratio for Temperature = 10 deg C in [%].

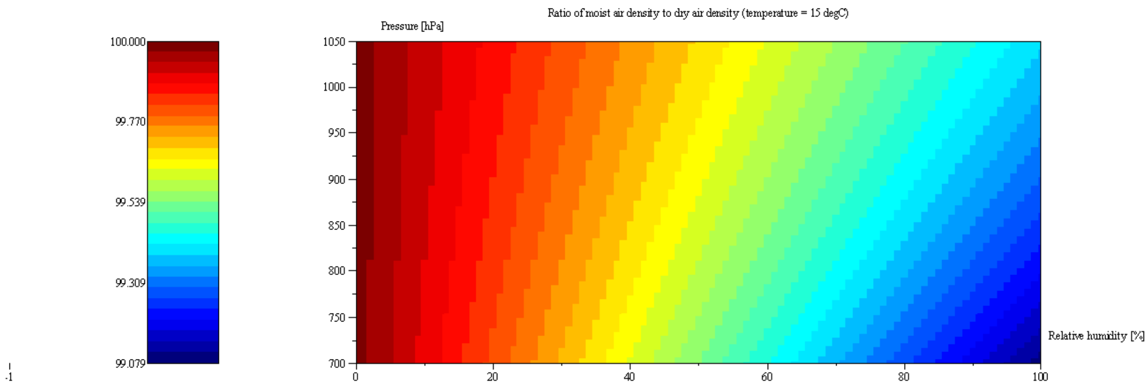


Figure 7: Density Ratio for Temperature = 15 deg C in [%].

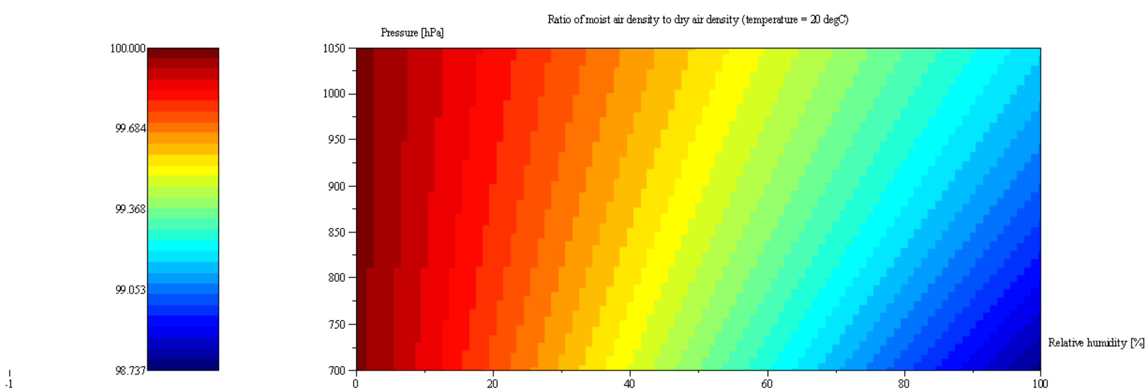


Figure 8: Density Ratio for Temperature = 20 deg C in [%].

References

[1] Murry L. Salby: *Fundamentals of Atmospheric Physics*, Academic Press, 1996, Elsevier Science
[2] *CRC Handbook of Chemistry and Physics*, CRC Press Inc., 60th edition, 1981,