Fitting Weibull Parameters for Wind Energy Applications

A number of different methods exist for estimating the distribution parameters in an arbitrary distribution. Among the most common methods are procedures using: 1) estimation of the statistical moments, 2) least squares methods, 3) maximum likelihood methods or 4) Bayesian methods. An introduction to these methods may be found in Ross [i]. Traditionally, it is recommended to use the maximum likelihood method for estimating the distribution parameters because of the ability of this method to estimate not only the parameters themselves but also a consistent estimate of any statistical uncertainty connected to the distribution parameters. When using the methods 1) or 2), it is possible to include uncertainty estimates on the distribution parameters using Bootstrap or Jackknife methods, see Efron and Tibshirani [ii].

1 The Weibull Distribution

For wind energy applications, the two-parameter Weibull distribution is used as the most common parametric distribution of the T-minutes averaged wind speed (often T=10 minutes):

$$F(u) = 1 - \exp\left[-\left[\frac{u}{A}\right]^k\right] \qquad (1)$$

$$f(u) = F'(u) = \frac{k}{A}\cdot\left[\frac{u}{A}\right]^{k-1} \exp\left[-\left[\frac{u}{A}\right]^k\right] \qquad (2)$$

where k is the Weibull form parameter
A is the Weibull scale parameter
$f(u)$ is the Weibull density function
$F(u)$ is the Weibull cumulative distribution function

The statistical moments, a_v, are related to the Weibull distribution parameters as follows:

Mean value: $$\mu = a_1 = A\cdot\Gamma[1+1/k] \qquad (3)$$

v-th order statistical moment: $$a_v = \int f(u)u^v du = A^v \cdot \Gamma[1+v/k] \qquad (4)$$

2 Estimating Moments in the Sample (Measured/Observed) Distribution

In the sample distribution, the statistical v-th order moments, a_v, are given by [iii, Chapter 27]:

$$a_v = \frac{1}{n}\sum_i x_i^v \qquad (5)$$

where a_v is the v-th order statistical moment
n is the number of samples
x_i is the sample with the index i

Note: The sample moments should be determined directly from the available time series/samples. However, it is also possible to estimate the sample moments from histograms – even if this procedure is slightly more inaccurate.

3 The Wind Power Density

The available wind power density is proportional to the mean cube of the wind speed. This may be expressed using the Weibull parameters:

$$\begin{aligned} E &= \int 0.5(\rho u)u^2 f(u)du = 0.5\rho a_3 \\ &= 0.5\rho A^3 \cdot \Gamma(1+3/k) \end{aligned} \qquad (6)$$

When dealing with the sample distribution, the wind power density is:

$$E_{sample} = 0.5\rho \frac{1}{n}\sum_i u_i^3 \qquad (7)$$

4 Estimating the Weibull Parameters for Wind Energy Applications

This section describes how to estimate the Weibull parameters using the method described in The European Wind Atlas [iv]. When using the 2-parameter Weibull distribution for wind energy applications, the requirements for the estimated 'optimum' distribution parameters may not be as specified in the four methods above. Indeed, The European Wind Atlas [iv] specifies two criteria quite different from the ones specified in the 'traditional' methods:

- The total wind energy in the fitted Weibull distribution and the observed distribution are equal.
- The frequency of occurrence of the wind speeds higher that the observed average speeds are the same for the two distributions.

These two requirements lead to an equation in *k* only. It is important to notice, that the average wind speed in the requirement in 2) is the sample wind speed. This leads to the required equation in *k* only. The equations (6) + (7) is used in order to determine the *A* parameter as a function of *k*:

$$\begin{aligned} &E = E_{sample} \Leftrightarrow \\ &A = \left[\left(\frac{1}{n}\sum_i u_i^3\right) \Big/ \Gamma(1+3/k)\right]^{1/3} = \left[a_3/\Gamma(1+3/k)\right]^{1/3} \end{aligned} \qquad (8)$$

Now, the requirement 2) is used with equation (8):

$$\begin{aligned} P[u > \mu_{sample}] &= 1 - F(\mu_{sample}) = \exp\left[-\left(\frac{\mu_{sample}}{A}\right)^k\right] \\ &= \exp\left[-\left(\frac{\mu_{sample}}{\left[a_3/\Gamma(1+3/k)\right]^{1/3}}\right)^k\right] \end{aligned} \qquad (9)$$

A standard root finding algorithm is utilized in order to determine *k* from eq. (9), as the probability of exceeding the mean value, $P[u>\mu_{sample}]$ is calculated directly from the sample distribution (time series or histograms). Now, when *k* is known (calculated from (9)), *A* is easily determined from (8), assuming that *E* or a_3 are also determined from the sample distribution.

Example – samples with histograms and time series data

Using time series data in a section between degrees = [255;285[from a Danish wind farm at Torrild, then the first and third order moments may be calculated to a_0= 5.593 m/s and a_3= 321.88 $(m/s)^3$ (using eq. 5). The probability of exceeding the mean wind speed is 0.45421 in the sample distribution. The *k* parameter may be determined to *k*=2.0444 and *A*=6.2801 m/s using equations (9) and (8).

Estimating the moments from a histogram using a 1 m/s bin width yields a_0= 5.654 m/s and a_3= 332.15 $(m/s)^3$ and the probability of exceeding the mean is 0.4511. The Weibull parameters can be determined to k=2.0267 and A=6.3275, i.e. a slightly different result that the one obtained using time series. The current implementation in WindPRO differs slightly from the specifications stated in the European Wind Atlas (using the distribution density function instead of the distribution function), but it also uses histograms in estimating the Weibull parameters. Here, an A=6.316 and k=2.0340 is found. WAsP reports A = 6.3 and k = 2.04. The histogram values that were used are stated below:

```
Wind  [m/s]  <1  2   3   4   5   6  7  8  9 10 11 12 13 14 15 16 17 18 19 20
Count [o/oo] 11 62 121 155 122 120 97 84 77 73 41 22  8  4  2  1  1  0  0  0
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Please note, that when calculating the mean value from the Weibull A and k parameters (see eq. (3)), then the mean value is slightly different from the one found directly from the sample values. This is due to the fact, that the fitting criteria assures only that the energy in the sample and fitted distributions are equal, and furthermore that the probability of exceeding the sample mean are the same. The means calculated from the Weibull A and k parameters are 5.564 m/s for the time series data and 5.606 m/s for the histogram data.

B.5 Estimating the uncertainty

One may utilize a bootstrap method [ii] to estimate the statistical uncertainty connected to the Weibull A and k parameters using the two requirements as shown above. Again dealing with the Torrild time series data, the results are as stated below. Here, the estimates on the statistical uncertainty are calculated using 10000 replications.

Sector	μ_A [m/s]	σ_A [m/s]	μ_k	σ_k
195-225 deg	6.7835	0.0588	2.2258	0.04748
225-255 deg	7.4052	0.0571	2.2861	0.04779
255-285 deg	6.2801	0.0545	2.0444	0.03675

In normal circumstances, the statistical uncertainty may be ignored because the uncertainty connected to the inherent variability (daily and seasonal variations) and the model uncertainty (e.g. WAsP) is magnitudes larger

[i] Ross, Sheldon M.: *Introduction to Probability And Statistics For Engineers and Scientists*, John Wiley & Sons, 1987.
[ii] Efron, Bradley & Robert J. Tibshirani: *An Introduction to the Bootstrap*, Chapman & Hall, 1998.
[iii] Cramér, Harald: *Mathematical Methods of Statistics*, Princeton University Press, 1946.
[iv] Troen, I & E.L. Petersen, *European Wind Atlas*, Risø National Laboratory, 1989, ISBN 87-550-1482-8.